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Optimal Vibration Control by the Use of Piezoceramic Sensors and Actuators

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In this paper, a discrete degrees of freedom model has been formulated for a structural dynamic system consisting of a linear elastic structure, bonded piezoceramic sensors and actuators, and a feedback signal conditioning system. In addition, an optimal control procedure based on the minimization of a quadratic performance index of state and control vectors has been developed that uses output feedback methods. Finally, the application of the model and the control technique has been demonstrated through the example of a linear elastic beam with piezoceramic sensors and actuators occupying discrete subdomains of the beam upper and lower surfaces. A model for the linear elastic beam has been obtained by using test results and a structural dynamic system identification method based on an equation error approach. Results for various weights in the performance index are included, and implications for future applications are discussed.

Introduction

IN the past few years, there has been considerable research activity in the field of active and passive control of vibrations of flexible structures. One of the methods of active control of vibrations, termed "electronic damping" in some of the early literature,[1-6] involves the placement of piezoceramic devices on a structure to sense and control dynamic strains induced by structural vibrations. The deformation of a sensing transducer results in an electrical current that is conditioned by operations such as amplification and shifting of the phase of the signal. The conditioned signal is then applied to another piezoceramic, electrostrictive, or magnetostrictive device placed at a selected location on the structure. This transducer acts as an actuator and transmits mechanical energy to the structure. Depending on the applied voltage, electromechanical coupling of the forcing transducer to the structure, and the location of the transducers, a degree of vibration control of flexible structures can be achieved. To date, applications of the aforementioned scheme have primarily been in the area of large space structures, such as in the work of Crawley and Deluis,[7] but the scheme is applicable to any structure with lightweight components.

This type of active control offers unique features that are not usually employed for control of structural vibrations. The dynamics of direct contact type sensors and actuators permit a wide frequency range of control. A measure of tunability is provided for the control of structural systems that age or grow. Finally, this method adds little mass to the controlled flexible structure so that the existing plant model does not need to be modified to account for the mass of the transducers.

To utilize the advantages of piezoceramic transducers, it is necessary to select appropriate positions of the transducers and to select the sensor signals that are to be fed back to the actuators. The problem of selecting the locations of the transducers is a complete problem in itself and thus will not be addressed in this paper. There has been some work in this area as can be seen in Refs. 8-11. It is well known that collocated sensors and actuators are advantageous from the viewpoint of stability.[12] However, in some applications noncollocated sensors and actuators give better performance.[13] In this paper, however, only collocated sensors and actuators will be considered.

The problem of selecting appropriate feedback gains is equally important to selecting transducer positions. In the early work with piezoceramic transducers, the gains were chosen basically by trial and error. Recently, Bailey and Hubbard[14] have discussed the problem of "instantly optimal" control of a structure using a single distributed PVDF actuator with a conventional accelerometer sensor. However, there are many issues that still need to be resolved before appropriate control design procedures are developed. Specifically, for a given piezoceramic material, electromechanical coupling, structure, and control domain, a designer should be able to obtain an optimum procedure for controlling the vibrations of the given system. To meet this design goal, the designer must be able to obtain a measure of the system changes in damping and stiffness due to the active feedback control similar to the way in which stiffness and mass properties are obtained by using other analytical techniques. These topics will be addressed in this paper.

The objective of this paper is to develop optimal control algorithms, based on quadratic performance indices, that are applicable to selected classes of flexible beam structures with piezoceramic sensors and actuators. Beam structures have

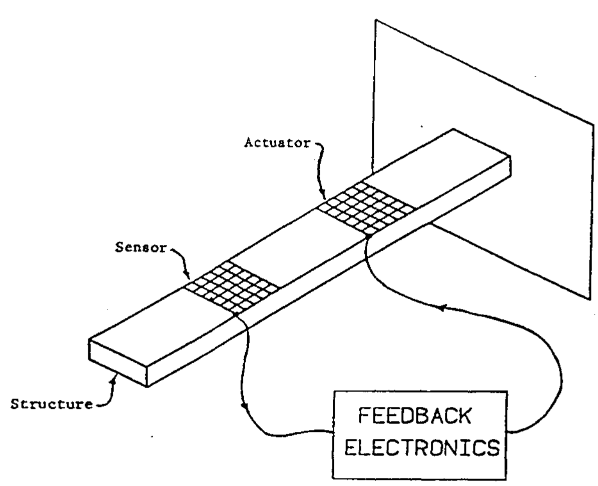

Fig. 1 Active control of structure with piezoelectric sensors and actuators.

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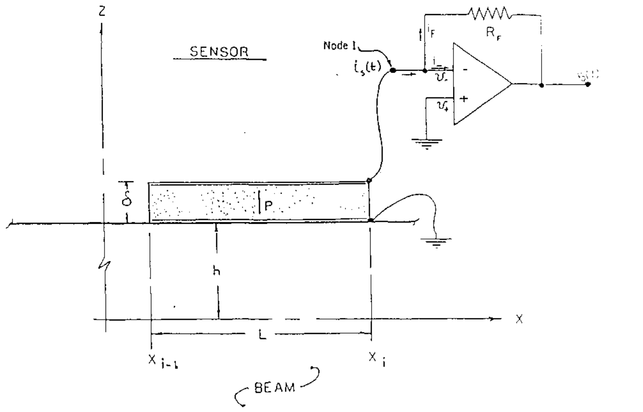

Fig. 2 Sensor with zero input impedance circuit.

been selected since most aerospace structures consist of beam bending members as the primary load-carrying elements. Thus, a model of an elastic beam with piezoceramic control and general boundary conditions will be developed here. New contributions in this paper include 1) the development of a discrete degrees of freedom model (finite element model) for an active beam with many distributed piezoceramic sensors and actuators coupled by an electrical conditioning system, and 2) the development of an active control algorithm using output feedback for a flexible structure equipped with sensors and actuators by using a quadratic performance index of state and control.

Finite Element Model for the Active Structure

In this section finite element models have been obtained for the individual components of an actively controlled structure (active structure) consisting of 1) a linear elastic beam, 2) surface-mounted piezoceramic sensors that occupy selected subdomains of the surface of the structure, 3) surface-bonded piezoceramic actuators that occupy different subdomains of the structure, and 4) the signal conditioning system (control electronics) that modifies the sensor output before feeding the signal to the actuators. A model for the active structure is then formulated by combining the different analyses. A sketch of the system to be analyzed is provided as Fig. 1. Previous work in this field has considered only the combination of a beam with piezoceramic actuators.

Sensor Analysis

An idealized sketch of a beam with a sensing transducer and a signal conditioning circuit is seen in Fig. 2. The piezoceramic transducer is polarized in the z direction and electrically connected via two short leads to a current amplifier. In the analysis, the following assumptions are made:

1) A perfect bond between the elastic body and the transducer has been assumed. In practice, there will be a relative displacement due to the deformation of the bonding material. This effect has been neglected.

2) The thin-film electrode surfaces have been assumed to add no mass or stiffness to the transducer. The transducer itself has been assumed not to add any mass or stiffness to the structure to which it is bonded.

3) The capacitance of the transducer leads have been considered to be negligible.

4) Temperature effects have been neglected. For cases in which these effects are important, the constitutive equations are usually modified.

One-dimensional constitutive equations for the piezoceramic material[15-18] are written as follows:

$$\begin{Bmatrix}\sigma_{11}\\ E_3\end{Bmatrix} = \begin{Bmatrix} C_{11} & -H \\ -H & \beta \end{Bmatrix}\begin{Bmatrix}\epsilon_{11}\\ D_3\end{Bmatrix} \tag{1}$$

or, conversely,

$$\begin{Bmatrix}\epsilon_{11}\\ D_3\end{Bmatrix} = \Delta\begin{Bmatrix} \beta & H \\ H & C_{11} \end{Bmatrix}\begin{Bmatrix}\sigma_{11}\\ E_3\end{Bmatrix} \tag{2}$$

where

$$\Delta = (\beta C_{11} - H^2)^{-1} \tag{3}$$

where σ_{11} is the stress, ϵ_{11} the strain, E_3 the electric field, D_3 the charge per unit area, C_{11} the elastic modulus under constant electric displacement, H the piezoelectric constant, and β the dielectric constant under constant strain. In the usual notations as proposed in the standards for piezoelectricity,[15] the stress σ_{11} is denoted by T_{11}, the dielectric constant β by E_{33}, and the constant H by h_{31}. For the type of piezoceramic crystal that is considered in this analysis, $\beta_{33} = \beta_{11} = \beta$.

The sensor thickness is an order of magnitude smaller than the elastic beam height $2h$. As such, the strain distribution through the sensor cross section is assumed to be constant and equal to the upper fiber strain of the beam at $z = \pm h$. When the sensor is deformed, both a charge and an electric field are produced in addition to a resulting stress in the piezoceramic material. By use of a zero input impedance circuit (Fig. 2), it is possible to show that the second terms on the right-hand sides of Eq. (2) are smaller than the first terms on the same side of the equation. Then the sensor ideally becomes a pure current source. Thus, for the transducer at $z = \pm h$,

$$\begin{aligned}\epsilon_{11} &= \Delta\beta\sigma_{11} = -hw''(x,t)\\ D_3 &= \Delta H\sigma_{11} = -\beta^{-1}hHw''(x,t)\end{aligned} \tag{4}$$

In these equations and the following discussions, ( )″ denotes partial differentiation with respect to the independent variable x. Now the input sensor current is proportional to the rate of charge developed. Then

$$i_s(t) = b^s\int_{x_{i-1}}^{x_i} \dot{D}_3\,\mathrm{d}x \tag{5}$$

In Eq. (5), b^s is the width of the transducer and the overdot represents partial differentiation with respect to time. The sensor has been assumed to occupy a subdomain $x_{i-1} < x < x_i$ and $z = h$. For the type of circuit considered in this analysis (Fig. 2), $i_\alpha = 0$, and the sensor output voltage becomes

$$v_0(t) = -R_f i_s(t) = k_s\int_{x_{i-1}}^{x_i} \dot{w}''(x,t)\,\mathrm{d}x \tag{6}$$

where

$$k_s = b^s\beta^{-1}hHR_f \tag{7}$$

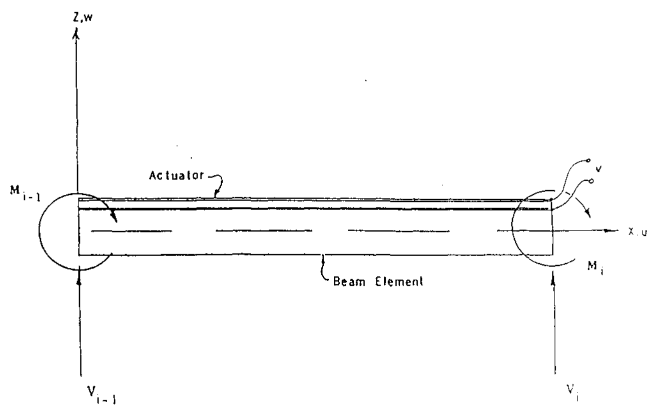

Fig. 3 Actuator beam element.

Actuator Analysis

With some changes, it is possible to use piezoceramic transducers as a sensor or an actuator. As a sensor, the transducers react to the deformations of the structure. When used as an actuator, a relatively high voltage on the order of 100–120 V rms is applied to the piezoceramic transducer plates. As a result of the converse piezoelectric effect, the transducers are deformed. Through the bonding of the transducer to the elastic structure, energy is transmitted to the elastic structure and the structure deforms. For purposes of analysis, it has been assumed that a positive voltage results in an elongation perpendicular to the polarization vector.

Assume that two unimorph piezoceramic actuators have been bonded to the surfaces of the beam at $z = \pm h$. The wiring is such that the actuator at $z = -h$ expands whenever the actuator at $z = -h$ contracts and vice versa. A finite element analysis of such a system has been reported by Kagawa and Gladwell.[16] In their analysis, Kagawa and Gladwell have not considered sensors or a feedback signal conditioning system. A brief summary of the analysis is given in the following.

The actuator is fully coated with electrodes. The wave propagation velocities associated with the electric field E_3 are very large compared to the linear elastic wave velocities of the beam. Therefore, the quantities v_d and E_3 are considered to be only functions of time. The charge D_3 developed in the actuator due to the applied voltage is a function of x and t. $D_3(x,t)$ is calculated by using Eq. (1) as follows:

$$v_d(t) = \int_h^{h+\delta} E_3(t)\,dz = \int_h^{h+\delta} (-H\epsilon_{11} + \beta D_3)\,dz \tag{8}$$

where δ is the thickness of the actuator. The piezoceramic transducer is a dielectric material that has been polarized in the z direction. Then

$$\frac{\partial D_3}{\partial Z} = 0 \qquad \text{or} \qquad D_3 = D_3(x,y,t) \tag{9}$$

It is assumed that there is no variation of D_3 in the z direction. Then

$$D_3(x,t) = (\beta\delta)^{-1} v_d(t) - \beta^{-1}(Hh + 0.5H\delta) w''(x,t) \tag{10}$$

By assuming $\delta \ll h$, the stress σ_{11} is calculated from Eq. (1):

$$\sigma_{11} = -H(\beta\delta)^{-1} v_d(t) - \beta^{-1}(Hh + 0.5H\delta) w''(x,t) \tag{11}$$

This expression is valid in the region $x_{i-1} < x < x_i$ and $h < z < h + \delta$. A similar expression can be written for the region $x_{i-1} < x < x_i$ and $-h - \delta < z < -h$ representing the transducer on the bottom surface of the beam. Stresses in the region $-h < z < h$ are given by

$$\sigma_{11} = -Ezw''(x,t) \tag{12}$$

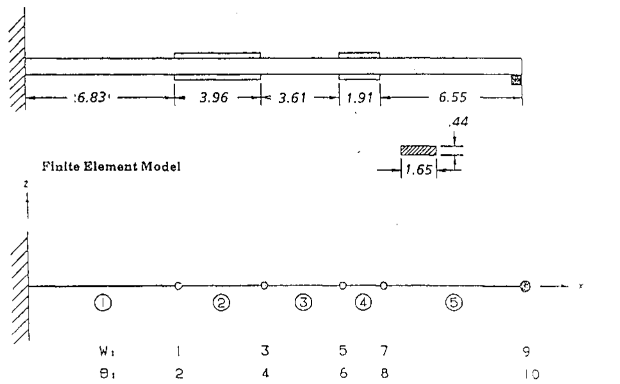

Fig. 4 Collocated sensor and actuator locations of the test beam.

where E is elastic modulus of the beam. Then an expression for the bending moment at a cross section $m^b(x,t)$ can be derived for the system with actuators at $z = +h$. By assuming $\delta \ll h$ and the usual sign convention for the positive bending moment,

$$m^b(x,t) = EIw''(x,t) - 2k_d v_d(t) \tag{13}$$

The flexural rigidity of the elastic beam is EI, and

$$k_d = \beta^{-1} H b^a h \tag{14}$$

where b^a is the width of the actuator. For one transducer only,

$$m^b(x,t) = EIw''(x,t) - k_d v_d(t) \tag{15}$$

From Eqs. (13) and (15), it is observed that a voltage applied to a piezoceramic actuator is equivalent to adding concentrated moments at locations at $x = x_i$ and $x = x_{i-1}$ denoting the boundaries of the actuator. The sign of this moment resulting from the applied voltage $v_d(t)$ can be changed by changing the sign of applied voltage. For a finite element analysis, the sign convention is illustrated in Fig. 3. The bending moment resulting from the applied voltage v_d adds to the positive bending moment at $x = x_{i-1}$, whereas it subtracts from the positive bending moment at $x = x_i$. Then the modified element load vector F^e is written as

$$F^e = K^e q^e \tag{16}$$

where

$$F^{eT} = \{V_{i-1},\ m^b_{i-1} + k_d v_d,\ V_i,\ m^b_i - k_d v_d) \tag{17}$$

$$q^{eT} = \{w_{i-1},\ \theta_{i-1},\ W_i,\ \theta_i) \tag{18}$$

$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6 & 12L^2 \\ -12 & 6 & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \tag{19}$$

Active Beam Element: Rate Feedback

Now consider a beam element containing both an actuator and a sensor. Assume that a piezoceramic transducer has been bonded to the surface at $z = +h$ and is used as a sensor. Similarly, assume that a piezoceramic transducer has been bonded to the surface at $z = -h$ and is used as an actuator. The output voltage from the sensor $v_s(t)$ is increased in amplitude by a constant amount G. The resulting voltage $Gv_s(t_s)$ is used as an input voltage to the actuator. The gain can be positive or negative, with the choice of sign depending on the orientation of the polarity vector of the unimorph transducers. Then, from the Eq. (6),

$$v_d(t) = Gv_s(t) = Gv_0(t) = Gk_s \int_{x_{i-1}}^{x_i} \dot{w}''(x,t)\,dx \tag{20}$$

By integrating and using the definitions

$$w'(x_i,t) = \theta_i; \qquad w'(x_{i-1},t) = \theta_{i-1} \tag{21}$$

Eq. (20) becomes

$$v_d(t) = Gk_s\left[\dot{\theta}(x_i,t) - \dot{\theta}(x_{i-1},t)\right] \tag{22}$$

Then the element force matrix takes the following form (assuming $G^* = -G$):

$$F^e = F^{e*} - C_{ED} q^e \tag{23}$$

where

$$F^{e*T} = \{V_{i-1},\ m_{i-1}^{b},\ V_i,\ m_i^{b}\} \tag{24}$$

and

$$C_{ED}^{e} = G^{*}k_D k_s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{25}$$

By introducing inertia forces, the complete dynamic equations with an internal viscous damping matrix are written as

$$M e \ddot{q}^{e} + C^{e}\dot{q}^{e} + C_{ED}^{e}\dot{q}^{e} + K^{e}q^{e} = F^{e*} \tag{26}$$

The damping matrix, which is the contribution of the rate feedback, can be called the "electronic damping" matrix[2] or the piezoelectric active damping matrix. These element matrices can be assembled in the usual way to yield the system equations of motion:

$$M\ddot{q} + C\dot{q} + Kq = F^{*} + F^{c} \tag{27}$$

where

$$F^{cT} = (-C_{ED}\dot{q})^{T} = \{0, M_1^{c}, 0, \ldots, -M_r^{c}\} \tag{28}$$

In these equations, F^c is the control force vector and M the mass matrix.

Active Beam Element: Modal Feedback

This phase of the analysis considers cases where the output signals from piezoceramic sensors have been conditioned by a signal conditioning system before feeding the signal to the corresponding actuator. In contrast to a rate feedback, the conditioned feedback has the potential of increasing stability by adding contributions to both the stiffness and damping matrices of the system.

A particular type of signal conditioning system that has been used by Forward and Swigert[5] has been considered here to illustrate the modeling procedure. For given input v_s, the output v_d from the signal conditioning system can be found from the equation

$$\ddot{v}_d + \frac{\omega_f}{Q}\dot{v}_d + \omega_f^2 v_d = \frac{G\omega_f}{Q} = \dot{v}_s \tag{29}$$

where ω_f is the filter center frequency, Q the bandwidth, and G the gain.

For r pairs of sensors and actuators, the equation can be expressed as

$$I\ddot{v}_d + A^{*}\dot{v}_d + B^{*}v_d = G^{h}\dot{v}_s \tag{30}$$

where

$$v_d^T = \{v_{D1}, \ldots, v_{Dr}\} \tag{31a}$$

$$v_s^T = \{v_{s1}, \ldots, v_{sr}\} \tag{31b}$$

The matrices A^* and B^* are diagonal matrices with diagonal elements:

$$A_{jj}^{*} = \omega_{fj}/Q_j; \qquad B_{jj}^{*} = \omega_{fj}^2 \tag{32}$$

Similarly,

$$G_{jj}^{h} = G_j\omega_{fj}/Q_j$$

Table 1 G-1195 piezoceramic coefficients[a]

Coefficient	Value	Units
Modulus	6.3×10^{10}	N/m^2
d_{31}	1.79×10^{-10}	m/V
e_{33}	1.65×10^{-8}	F/m
g_{31}	1.14×10^{-2}	V-m/N
Density	7.6×10^{0}	g/cm^3

[a] IEEE standard notations. Equivalent constants C_{11}, $\bar{k}$, H, and B can be calculated.

By using Eq. (22) to relate v_{dj} of each actuator to the corresponding sensor output v_{sj} and using a sensor location matrix T_s to assemble at all the sensor-actuator locations, Eq. (30) is rewritten as follows:

$$I\ddot{v}_d + A^{*}\dot{v}_d + B^{*}v_d = K_sT_s\ddot{q} \tag{33}$$

where K_s is a diagonal matrix with diagonal elements

$$K_{sjj} = (G_j\omega_{fj}/Q_j)k_{sj} \tag{34}$$

The force vector F^e from Eq. (16) is now rewritten as follows:

$$F^{e} = F^{e*} + K_d^{e}v_d^{e} \tag{35}$$

where

$$K_d^{e} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} k_d \tag{36}$$

$$v_d^{eT} = \{0,\ v_{di},\ 0,\ v_{di}\} \tag{37}$$

By use of an actuator location matrix T^D, the coupled set of finite element equations for an active beam system with signal conditioning feedback and piezoceramic sensors and piezoceramic actuators are

$$I\ddot{v}_d + A^{*}\dot{v}_d + B^{*}v_d = K_sT_s\ddot{q}$$

$$M\ddot{q} + C\dot{q} + Kq = T^{D}K_D\dot{v}_d + F^{*} \tag{38}$$

Equations (38) represent a coupled set of ordinary differential equations. Equations similar to these have been derived by Hallauer et al.[19] for a structure with accelerometer sensors and shaker type of actuator.

Optimal Control of Flexible Structures with Piezoelectric Sensors and Actuators

In the preceding section, finite element models have been developed for a structural dynamic system consisting of a linear elastic beam, piezoceramic sensors, piezoceramic actuators, and output feedback. Both rate feedback and signal conditioning have been considered. In this section these models are used to develop an optimal control procedure. In the first phase of the analysis, only rate feedback is considered. The control objective is to bring the structure from a perturbed state to a state near the initial state and determine the associated gains by minimizing a quadratic performance index of the state and control variable.

In most practical problems involving the control of flexible structures, similar to the system considered in this paper, it is not possible to measure the complete state. The control

Table 2 Numerical values for the rate control analysis model

$$[A]=\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2.334e7 & 3.465e7 & -1.938e7 & 2.734e6 & 1.097e6 & -24.42 & 14.05 & 1.241 & -8.169 & -1.856 \\ 5.653e7 & -1.204e8 & 1.180e8 & -4.236e7 & -4.915e6 & 24.42 & -47.55 & 19.19 & -1.528 & 2.264 \\ -3.016e7 & 1.109e8 & -2.429e8 & 1.656e8 & -1.290e7 & 5.013 & 21.73 & -66.56 & 32.01 & 4.129 \\ -4.608e6 & -1.090e7 & 1.425e8 & -1.799e8 & 5.264e7 & 3.169 & 4.390 & 26.19 & -51.46 & 13.49 \\ 6.401e6 & -6.459e6 & -9.703e7 & 1.772e8 & -7.933e7 & -5.384 & 7.757 & -2.545 & 37.50 & -43.32 \end{bmatrix}$$

$$[B]=\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \\ -1503.0 & -3937.0 \\ 7544.0 & 2.087e4 \\ -1.475e4 & -4.326e4 \\ 1.006e4 & 3.130e4 \\ -8.576e4 & -2.738e4 \end{bmatrix} \qquad [C]=\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -407.4 & 407.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -407.4 & 407.4 & 0.0 \end{bmatrix}$$

vector is defined to be proportional to an output vector. Such a system is usually defined in the following manner:

$$\dot{x} = Ax + Bu, \qquad x \in R^n \tag{39a}$$

$$y = Cx, \qquad y \in R^n \tag{39b}$$

$$u = -Gy \tag{39c}$$

where A, B, C, and G are constant coefficient matrices. The first step in the analysis is to reduce the developed finite element model to the form of Eqs. (39).

Transformation of the Active Beam Equations

The beam has been assumed to be in a perturbed state and free of external forces and moments. Then the external force vector F^* is zero. The control force vector F^c has only nonzero moments that have been produced by an output feedback voltage applied to the actuators. All shear force components in the control force vector F^c are zero. Then the next step is to eliminate the translational degrees of freedom w_i corresponding to zero shear forces in Eqs. (28) and (29). This is accomplished by using a suitable condensation procedure. In this case,

$$M^r\ddot{\theta} + C^r\dot{\theta} + K^r\theta = F^c \tag{40}$$

where

$$\theta^T = \{\theta_1, \ldots, \theta_n\} \tag{41a}$$

$$F^{cT} = \{-m_1^b, \ldots, m_n^b\} \tag{41b}$$

The quantities m_j^b are the control moments. These moments are present at all nodes corresponding to the ends of active actuators. The moments m_j^b depend on the sensor output v_{si}, gain G_j, and constants k_s and k_d associated with different sensors and actuators.

For a beam with n nodes and r sensors, sensor output voltages are represented by an $r \times 1$ vector:

$$v_s = K_s\dot{\theta} \tag{42}$$

The matrix K_s is an $r \times n$ sensor location matrix, and $\dot{\theta}$ is an $n \times 1$ angular velocity vector. For a system with two sensors and five nodes (Fig. 4), K_s becomes

$$K_s = \begin{bmatrix} -k_{s1} & k_{s1} & 0 & 0 & 0 \\ 0 & 0 & -k_{s2} & k_{s2} & 0 \end{bmatrix} \tag{43}$$

Then the control moments corresponding to r collocated actuator configurations are obtained by the following equation:

$$m^b = K_D G K_s \dot{\theta} \tag{44}$$

In Eq. (44), this equation m^b is an $n \times 1$ vector with zeros at appropriate nodes corresponding to the absence of actuators. The matrix G is an $r \times r$ matrix of constant gains, and the matrix K_D is an $n \times r$ actuator location matrix. For a system with two sensors, two actuators, and n nodes (Fig. 4),

$$G = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \quad \text{or} \quad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{45}$$

and

$$K_D = \begin{bmatrix} k_{D1} & 0 \\ -k_{D1} & 0 \\ 0 & k_{D2} \\ 0 & -k_{D2} \\ 0 & 0 \end{bmatrix} \tag{46}$$

The gain matrix is diagonal when a sensor output is fed back to the corresponding collocated actuator. The gain matrix is fully populated when output from each sensor is fed back to an actuator with different gains g_{ij}. Equation (40) is now rewritten as follows:

$$\theta = -M^{r-1}C^r\dot{\theta} - M^{r-1}K^r\theta + M^{r-1}K_D G K_s \dot{\theta} \tag{47}$$

At this stage, the following variables have been introduced:

$$x^T = \{x_1, \ldots, x_{2n}\} = \{\theta_1, \ldots, \theta_n, \dot{\theta}_1, \ldots, \dot{\theta}_n\} \tag{48}$$

and

$$y^T = v^T \tag{49}$$

Then the equations are reduced to the forms of Eqs. (39) with

$$A = \begin{bmatrix} 0 & I \\ -M^{r-1}K^r & -M^{r-1}C^r \end{bmatrix} \tag{50}$$

$$B = \begin{bmatrix} 0 \\ -M^{r-1}K_D r \end{bmatrix} \tag{51}$$

and

$$C = [0 \quad K_s] \tag{52}$$

Optimal Control

The control objectives are to select the elements of the gain matrix G by minimizing a quadratic performance index in the state and control variables:

$$J = E_{x_0}\left\{\int_0^\infty (x^TQx + u^TRu)\,\mathrm{d}t\right\} + \gamma(G) \tag{53}$$

where the expectation is taken over a presumed distribution on x_0. In the following, this distribution is taken so that $E\{x_0\} = 0$; $E\{x_0 \ x_0^T\} = X_0$. The quantity $\gamma(G)$ is any scalar function having continuous gradients in G. The positive-definite matrices Q and R determine the relative weights of the state and control variables at different locations. Similar optimization problems have been studied by Levine and Athans[20] and later by Mendel.[21] A convergent solution algorithm has been developed by Moerder and Calise.[22] The function $\gamma(G)$ is useful in penalizing the off-diagonal terms of the gain matrix G to ensure their proximity to zero.

A brief summary of the numerical procedure is explained here. It is well known that the minimization of J in Eq. (53) is equivalent to minimizing

$$J = \mathrm{tr}(K_c X_0) + \gamma(G) \tag{54}$$

where K_c satisfies the equation

$$S_c(G,K_c) = (A - BGC)^T K_c + K_c(A - BGC) + Q - C^TGRGC = 0 \tag{55}$$

It has been shown that the minimization process reduces to the solution of the following equations:

$$(A - BGC)^T L + L(A - BGC) + X_0 = 0 \tag{56a}$$

$$S_c(G,K_c) = 0 \tag{56b}$$

$$G = R^{-1}\left[B^T K_c L C^T - \gamma_G(G)/2\right](CLC^T)^{-1} \tag{56c}$$

for the unknowns G, K_c, and L, where $\gamma_G(G) = \partial\gamma/\partial G$. L is a matrix of Lagrange multipliers used to enforce the constraint (55) in the optimization process.

A numerical solution for these equations can be obtained by first specifying the matrices A, B, C, Q, R, and X_0 and providing an initial guess for the unknown gain matrix G such that $(A - BGC)$ is stable. This initial guess is used in the first iteration ($j = 1$) to obtain the values of the matrices L_j and K_{cj} from Eqs. (56a) and (56b). Next, Eq. (56c) is used to define

$$\Delta G_j = R^{-1}\left[B^T K_{cj} L_j C^T - \gamma(G)/2\right](CL_jC^T)^{-1} - G_j \tag{57}$$

Then the elements of the gain matrix G_{j+1} for the next iteration are obtained as follows:

$$G_{j+1} = G_j + \alpha \Delta G_j \tag{58}$$

A value of α between 0 and 1 is chosen at each iteration to ensure that $(A - BG_{j+1}C)$ is stable and that the value of $J(G_{j+1})$ in Eq. (54) is less than the value of $J(G_j)$. The proof of convergence[22] guarantees both the existence of α so long as G_j is not at a stationary point and that the process converges to a local minimum for J. However, more than one locally minimizing solution may exist.

Table 3 Diagonal values of Q[a]

Element	$Q^{(a)}$	$Q^{(b)}$	$Q^{(c)}$
$q_{1,1}$	1.0	1.0	0.0
$q_{2,2}$	6.218×10^{-4}	2.4936×10^{-2}	0.0
$q_{3,3}$	1.0062×10^{-5}	3.1720×10^{-3}	0.0
$q_{4,4}$	6.5785×10^{-5}	8.1108×10^{-3}	0.0
$q_{5,5}$	9.8778×10^{-8}	3.1429×10^{-4}	0.0
$q_{6,6}$	1.0	1.0	1.0
$q_{7,7}$	6.2181×10^{-4}	2.4936×10^{-2}	1.0
$q_{8,8}$	1.0062×10^{-5}	3.1720×10^{-3}	1.0
$q_{9,9}$	6.5785×10^{-5}	8.1108×10^{-3}	1.0
$q_{10,10}$	9.8778×10^{-8}	3.1429×10^{-4}	1.0

[a]Off-diagonal terms were zero.

Table 4 Optimal gains for $R = I$ and Q from Table 3

Matrix	Elements	
$G^{(a)}$	-6.3535×10^{-5}	-2.6578×10^{-4}
	-1.1942×10^{-4}	8.5920×10^{-4}
$G^{(b)}$	-6.5327×10^{-5}	-2.8128×10^{-4}
	-1.1902×10^{-4}	9.3083×10^{-4}
$G^{(c)}$	-3.2897×10^{-5}	-2.8214×10^{-4}
	-4.7534×10^{-4}	1.0741×10^{-4}

Numerical Results

An example of a cantilever beam has been considered to illustrate the developed procedure for optimal vibration control of structures by the use of piezoceramic sensors, actuators, and rate feedback. The cantilever beam is 22.86 cm long and has cross-sectional dimensions of 1.65×0.44 cm. The beam is made of an aluminum alloy. Two piezoceramic transducers made of lead zirconate titanate (G1195) of sizes $1.91\times1.65\times0.0254$ cm and $3.96\times1.65\times0.0254$ cm have been selected for use as collocated sensors and actuators. As is the state of the art, the formulated finite element model does not contain the values for the damping matrices. A linear viscous damping matrix has been determined from tests conducted on the beam using a structural system identification procedure that assumes linear viscous damping. The first 10 eigenvalues, eigenvectors, and an a priori model are required in the use of the selected identification procedure,[23] which is based on an equation error approach. The derived finite element model has been used as an a priori model. Laboratory tests have been conducted, and the required eigendata have been obtained by using a GENRAD computer-aided data acquisition system and SDRC modal plus software. The identified model resulting from the identification algorithm yields the experimentally obtained eigendata and a symmetric damping matrix. This damping

Table 5 Optimal gains for $Q^{(a)}$ from Table 3 with penalized and fully supressed off-diagonal terms

Matrix	Elements	
$G_p^{(a)}$	-3.2306×10^{-7}	-6.9850×10^{-8}
	-1.7184×10^{-10}	2.9212×10^{-3}
$G_s^{(a)}$	-3.2306×10^{-7}	0.0
	0.0	2.9212×10^{-3}

Table 6 Comparison of changes in frequency and damping values with active control ($G^{(a)}$)

	No control		With control, $G^{(a)}$	
DOF	ω	ζ	ω	ζ
1	375.98	0.0178	377.98	0.1098
2	2330.2	0.0041	2357.6	0.1607
3	6535.8	0.0038	6866.8	0.0402
4	12,919.0	0.0022	12,603	0.0208
5	20,753.0	0.0023	19,906	0.5096

Table 7 Frequency and damping values for the penalized and suppressed off-diagonal terms (diagonal G cases)

	$G_p^{(a)}$		$G_s^{(a)}$	
DOF	ω	ζ	ω	ζ
1	376.00	0.0181	375.98	0.0181
2	2330.2	0.0044	2330.2	0.0044
3	6536.2	0.0037	6536.1	0.0036
4	12,920.0	0.0025	12,920.0	0.0025
5	20,750.0	0.2159	20750.0	0.2159

matrix has been denoted as the baseline matrix in the paper to distinguish it from the augmented damping matrix due to an active control input to piezoceramic actuators.

Piezoceramic properties of the selected sensors and actuators are listed in Table 1. The matrices A, B, and C of Eqs. (39) for the cantilever beam were obtained from the identified mass, stiffness, and damping matrices and are provided as Table 2. In the process of obtaining matrices A, B, and C, five translational degrees of freedom have been eliminated using a guyan condensation technique. The function $\gamma(G)$ has been selected to be

$$\gamma(G) = \nu(g_{12}^2 + g_{21}^2)/2 \tag{59}$$

to reduce the off-diagonal terms. Three different types of weights on the states were selected and are illustrated in Table 3. In case a, the diagonal elements are inversely proportional to the square of the eigenvalues, whereas in case b they are inversely proportional to the eigenvalues. In the third case the state rates are equally penalized. For all cases, both R and X_0 were chosen to be identity matrices. Optimal gains were obtained for cases in which off-diagonal gain terms are penalized and off-diagonal terms are not penalized. The latter case corresponds to the case where each sensor output has been fed back to both actuators with appropriate gains. The fully populated gain matrices for Q corresponding to each of the cases in Table 3 are listed in Table 4. Table 5 illustrates what happens when the off-diagonal terms are penalized ($\nu \gg 0$) and then set to zero. In Tables 6 and 7, values of frequency and damping with and without control are listed.

Figure 5 is the time history of the tip velocity for the open-loop system. Sensor outputs were similar. Figure 6 is the corresponding closed-loop time history of tip velocity when off-diagonal terms have been penalized. The closed-loop time history of tip velocity for cases where off-diagonal terms have been penalized and then set to zero after the optimization process are essentially identical. Figure 7 shows the closed-loop time history for the tip velocity for systems with cross feedback, where off-diagonal terms are not penalized. Finally, Fig. 8 is the input to actuator 1 for this case of cross feedback. The input to actuator 2 was similar.

Discussion

From an examination of Tables 2, 4, and 5, it is obvious that the choice of weighting matrix Q plays a key role. The choice of Q affects 1) the number of iterations needed to converge to an optimal solution of the gain matrix G, and 2) the magnitudes of the resulting gain and the consequent changes in the magnitudes of the damping coefficients (in each mode) that can be achieved by the use of rate feedback and piezoelectric sensors and actuators. It has been found that the use of identity matrices for Q results in poor convergence and very little change in the damping constants from baseline damping constants. Accordingly, the time taken to return a given perturbation to the initial state is large. However, a Q matrix with diagonal matrix elements equal to the inverse of the squares of the eigenvalues required a relatively lower number of iterations to converge and resulted in large changes in damping coefficients, particularly in the first and fifth modes.

A comparison of time histories with and without penalty for off-diagonal terms suggests the superiority of a system with cross feedback, where each sensor output has been fed back to both actuators. Practical implementation, however, still needs to be studied. Setting the off-diagonal gain terms to zero after optimization (with a large penalty on these gains) did not significantly change the results. All diagonal gain matrices exhibit significantly lower damping. It is also observed from the time histories that the first mode takes a significantly larger amount of time to attenuate in comparison to other modes.

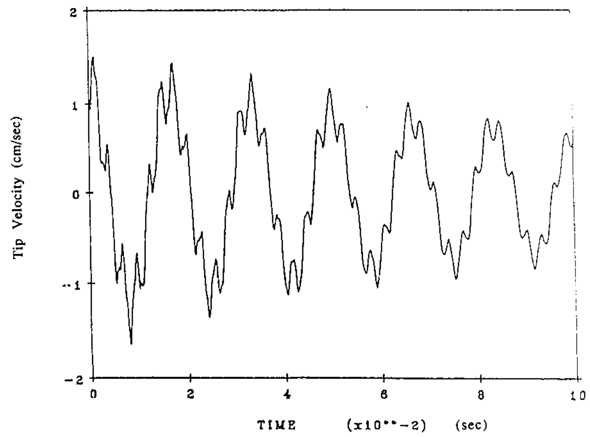

Fig. 5 Time history of tip velocity, no control.

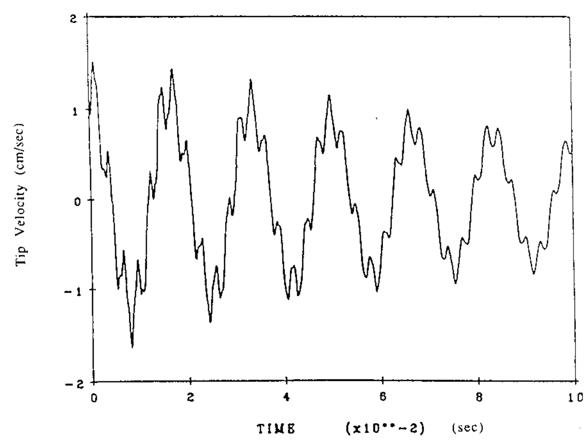

Fig. 6 Time history of tip velocity for penalized gain matrix.

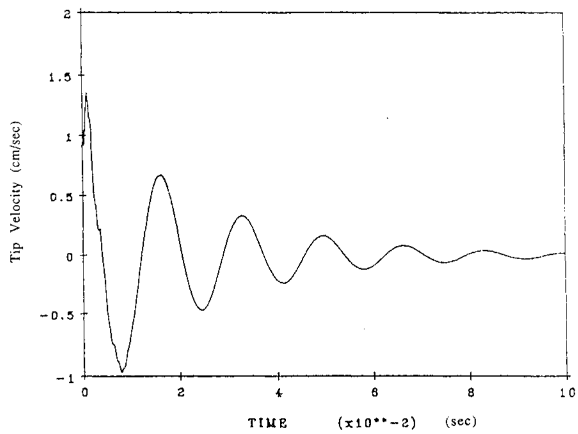

Fig. 7 Closed-loop time history of tip velocity for the general gain matrix with cross feedback.

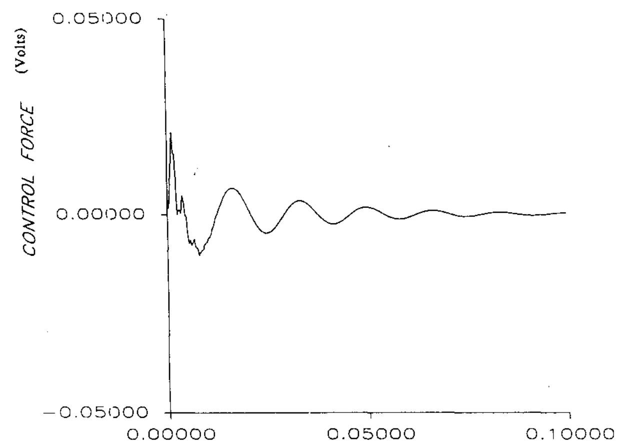

Fig. 8 Control input for actuator 1 for the general gain matrix with cross feedback.

This can be attributed to a large amount of energy in the first mode in comparison to the other modes.

This study also demonstrates that it is possible in principle to increase damping in several modes by using two piezoceramic sensors and actuators. Of course, practical aspects of this study including the effects of spillover still need to be investigated.

Conclusions

It has been shown that models with discrete degrees of freedom can be developed for active structural dynamic systems consisting of piezoceramic sensor, actuators, and an output feedback with signal conditioning. It has also been demonstrated that optimal control procedures can be developed by minimizing a quadratic index of the state and control. The limited numerical study has revealed that, in practical problems requiring a specific nature of the output or time targets, further study or different procedures for optimal control are needed. Future studies should not prespecify the signal conditioning. Instead, dynamic compensators with frequency weighted cost functions, or an H_∞ design approach, would be more appropriate. However, the present example does demonstrate a case in which cross feedback leads to a superior control and thus indicates the importance of also optimizing the location of sensors and actuators in any application.

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